

Closing Wed: HW_8A, 8B (8.3, 9.1)

Closing Next Wed: HW_9A, 9B (9.3, 9.4)

9.1 Intro to Differential Equations

A **differential equation** is an equation involving derivatives.

A **solution to a differential equation** is any function that satisfies the equation.

Entry Task:

Find $y = y(x)$ such that

$$\frac{dy}{dx} - 8x = x^2 \text{ and } y(0) = 5.$$

Check your final answer

$$\frac{dy}{dx} = 8x + x^2$$

$$\Rightarrow y = 4x^2 + \frac{1}{3}x^3 + C$$

$$y(0) = 5 \Rightarrow 5 = 4(0)^2 + \frac{1}{3}(0)^3 + C$$

$$\Rightarrow C = 5$$

$$y = 4x^2 + \frac{1}{3}x^3 + 5$$

CHECK:

$$\bullet y(0) = 5 \quad \checkmark$$

$$\frac{dy}{dx} = 8x + x^2$$

$$\bullet \text{LHS} \stackrel{\text{so}}{=} \frac{dy}{dx} - 8x = (8x + x^2) - 8x = x^2 = \text{RHS} \quad \checkmark$$

Example

Consider the differential equation:

$$\frac{dP}{dt} = 2P$$

(a) Is $P(t) = 8e^{2t}$ a solution?

$$\text{LHS} = \frac{dP}{dt} = 16e^{2t} \quad \leftarrow \text{ALWAYS SAME!}$$
$$\text{RHS} = 2P = 16e^{2t} \quad \leftarrow$$

YES!!

(b) Is $P(t) = t^3$ a solution?

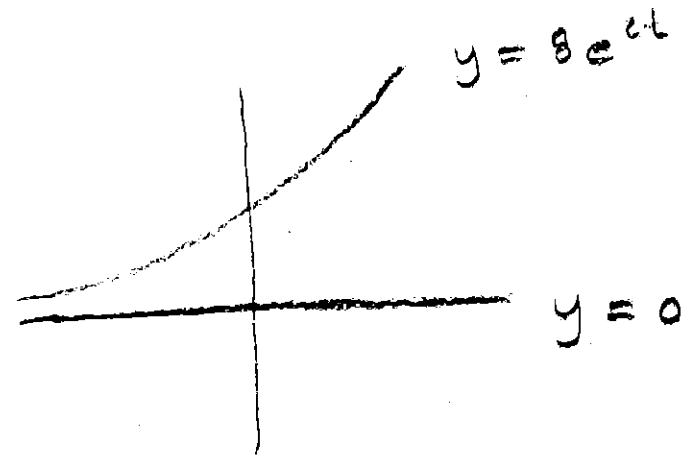
$$\text{LHS} = \frac{dP}{dt} = 3t^2 \quad \leftarrow \text{NOT ALWAYS THE SAME!}$$
$$\text{RHS} = 2P = 2t^3 \quad \leftarrow$$

NO!

(c) Is $P(t) = 0$ a solution?

$$\text{LHS} = \frac{dP}{dt} = 0 \quad \leftarrow \text{ALWAYS THE SAME}$$
$$\text{RHS} = 2P = 0 \quad \leftarrow$$

YES!



The **general solution** to

$$\frac{dP}{dt} = 2P$$

s

$$P(t) = Ce^{2t},$$

for any constant C .

We will learn how to find this next time.

Example: Consider the 2nd order differential equation

$$y'' + 2y' + y = 0.$$

(a) Is $y = e^{-2t}$ a solution?

$$y' = -2e^{-2t}, \quad y'' = 4e^{-2t}$$

$$\text{LHS} = (4e^{-2t}) + 2(-2e^{-2t}) + e^{-2t} = e^{-2t}$$

$$\text{RHS} = 0$$

NO

← NOT THE SAME

(b) Is $y = t e^{-t}$ a solution?

$$y' = -t e^{-t} + e^{-t} = e^{-t}(1-t)$$

$$y'' = -e^{-t}(1-t) - e^{-t} = e^{-t}(t-2)$$

$$\text{LHS} = e^{-t}(t-2) + 2e^{-t}(1-t) + t e^{-t}$$

$$= e^{-t}(t-2+2-2t+t)$$

$$= 0$$

$$\text{RHS} = 0$$

YES!

(c) There is a sol'n that looks like

$$y = e^{rt}.$$

Can you find the value of r that works?

$$y' = r e^{rt}$$

$$y'' = r^2 e^{rt}$$

$$\text{LHS} = y'' + 2y' + y = r^2 e^{rt} + 2r e^{rt} + e^{rt} \stackrel{?}{=} 0 \quad \leftarrow \text{RHS}$$

$$e^{rt}(r^2 + 2r + 1) \stackrel{?}{=} 0$$

$$e^{rt}(r+1)^2 \stackrel{?}{=} 0$$

$$\boxed{r = -1}$$

$$\boxed{y = e^{-t}} \quad \text{IS A SOL'N.}$$

Application Notes:

$\frac{dy}{dt}$ = "instantaneous rate of change of y with respect to t "

'A is proportional to B' means
 $A = kB$, where k is a constant.
In other words, $A/B = k$.

Ex)

| A | B |
|------|-----|
| 10 | 2 |
| 100 | 20 |
| 1000 | 200 |

$$A = k B$$

↑ PROPORTIONALITY CONSTANT

Ex)

| P | B |
|----------------------|-----------------------|
| Population of a town | Babies born in a year |
| 1000 | 40 |
| 10,000 | 400 |
| 100,000 | 4000 |

$$B = k P$$

40 ↑ ↑ 1000

⇒ $k = 0.04$ ← "relative" growth constant

(POPULATION ROUGHLY GROWS)
(AT 4% PER YEAR)

Some examples:

I. Natural Unrestricted population

Assumption: *"The rate of growth of a population is proportional to the size of the population."*

$P(t)$ = the population at year t ,
 $\frac{dP}{dt}$ = the rate of change of the population with respect to time (i.e. rate of growth).

So the assumption is equivalent to

$$\frac{dP}{dt} = kP,$$

or some constant k .

2. Newton's Law of Cooling

Assumption: "The rate of cooling is proportional to the temperature difference between the object and its surroundings."

T_s = constant temp. of surroundings

$T(t)$ = temp. of the object at time t ,

$\frac{dT}{dt}$ = rate of change of temp. with respect to time (i.e. cooling rate).

$T - T_s$ = temp. difference between object and surroundings.

So Newton's Law of Cooling is equivalent to

$$\frac{dT}{dt} = k(T - T_s),$$

or some constant k .

Ex)

| T | T_s | A |
|----------------|----------------------|------------------------------|
| Temp of Coffee | Temp of Surroundings | CHANGE IN TEMP IN ONE MINUTE |
| 170° | 70° | -2° |
| 150° | 70° | -1.6° |
| 120° | 70° | -1° |
| 100° | 70° | -0.6° |
| 80° | 70° | -0.2° |

$$B = T - T_s$$

$$A = k(T - T_s)$$

↑
-0.02

3. A Mixing Problem

Assume a 50 gallon vat is initially full of pure water.

A salt water mixture is being dumped **into** the vat at 2 gal/min and this mixture contains 3 grams of salt per gal. The vat is thoroughly mixed together.

At the same time, the mixture is coming **out** of the vat at 2 gal/min.

Let $y(t)$ = grams of salt in vat at time t .

$\frac{y(t)}{50}$ = salt per gallon in vat at time, t .

$\frac{dy}{dt}$ = the rate (g/min) at which salt is changing with respect to time.

Thus,

$$\text{RATE IN} = \left(3 \frac{\text{g}}{\text{gal}} \right) \left(2 \frac{\text{gal}}{\text{min}} \right) = 6 \frac{\text{g}}{\text{min}}$$

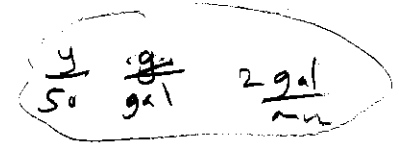
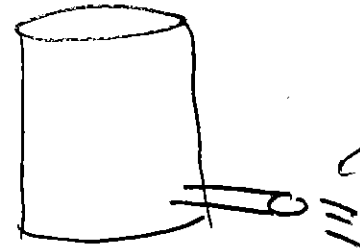
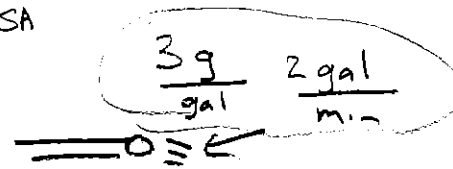
SALT CONCENTRATION
INCOMING MIXTURE

$$\text{RATE OUT} = \left(\frac{y}{50} \frac{\text{g}}{\text{gal}} \right) \left(2 \frac{\text{gal}}{\text{min}} \right) = \frac{y}{25} \frac{\text{g}}{\text{min}}$$

Thus,

$$\frac{dy}{dt} = 6 - \frac{y}{25}$$

SA



4. All motion problems!

Consider an object of mass m kg moving up and down on a straight line.

Let $y(t)$ = 'height at time t '

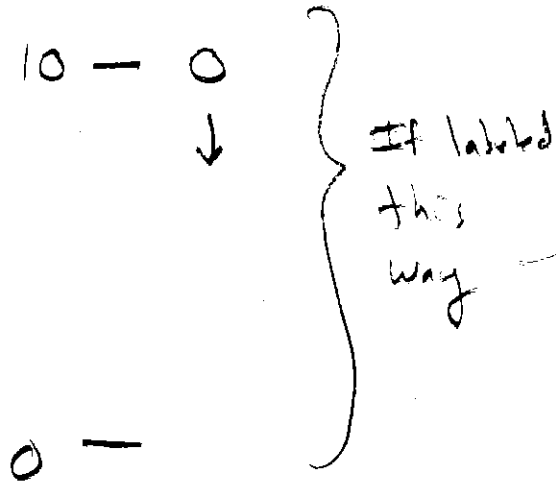
$\frac{dy}{dt}$ = 'velocity at time t '

$\frac{d^2y}{dt^2}$ = 'acceleration at time t '

Newton's 2nd Law says:

(mass)(acceleration) = Force

$m \frac{d^2y}{dt^2}$ = sum of forces on the object



Only taking into account gravity we get

$$m \frac{d^2y}{dt^2} = -mg$$

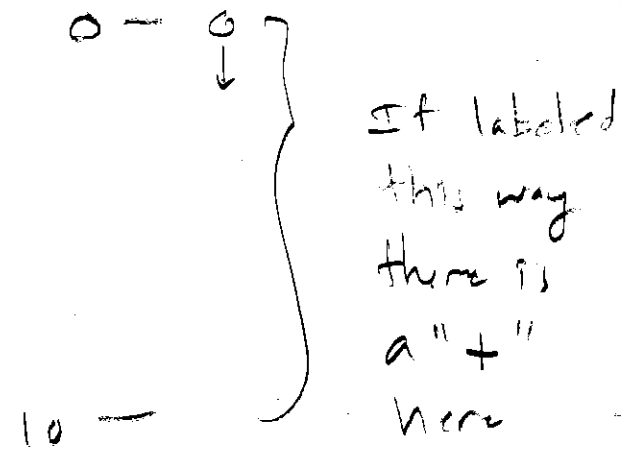
Now consider gravity and **air resistance**.

One of the most common models is to assume the force due to air resistance is proportional to velocity and in the opposite direction of velocity.

Then we get

$$m \frac{d^2y}{dt^2} = -mg - k \frac{dy}{dt}$$

ALWAYS A "-"
HERE



5. Many, many others:

Example:

A common assumption for melting snow/ice is "the rate at which the object is melting (rate of change of volume) is proportional to the exposed surface area."

Consider a melting snowball:

$$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$$

Write down the differential equation for r .

$$\frac{dV}{dt} = k S$$

$$\frac{4}{3}\pi 3r^2 \frac{dr}{dt} = k \cdot 4\pi r^2$$

$$\Rightarrow 4\pi r^2 \frac{dr}{dt} = 4\pi k r^2$$

$$\Rightarrow \boxed{\frac{dr}{dt} = k}$$

RADIUS CHANGING
AT A CONSTANT
RATE!